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ON SOME POINTS IN THE THEORY OF THE HYPERGEOMETRIC FUNCTION EXPRESSED AS A DOUBLE CIRCUIT INTEGRAL.

BY R. M. HATHAWAY.

IT is well known that the hypergeometric series

$$F(a, b, c, x) \equiv 1 + \frac{a \cdot b}{1 \cdot c} x + \frac{a(a+1)b(b+1)}{1 \cdot 2\, c(c+1)} x^2 + \cdot \cdot \cdot$$

satisfies the differential equation*

$$x(1-x) \frac{d^2y}{dx^2} + [c-(a+b+1)x] \frac{dy}{dx} - aby = 0.$$

If any solution of this differential equation be called a *hypergeometric function*, then the definite integral

$$\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-xu)^{-a}du$$

is, whenever it has a meaning, such a function, since it satisfies the differential equation. When, however, the exponents $b-1$ and $c-b-1$ are such that the real part of b is not greater than zero, or the real part of $c-b$ is not greater than zero, the integral ceases to converge, and it is necessary to seek some other expression for the hypergeometric function. The difficulty arises from the fact that the path of integration goes through two of the singular points of the function to be integrated. If, then, the path of integration be so chosen that the singular points are avoided, the integral will always converge, whatever may be the values of the exponents. In order that the function thus defined satisfy the hypergeometric differential equation, it is sufficient that this path be a closed one and that the function to be integrated shall have the same value at the end of the path as at the beginning; in other words, that the path of integration be a closed path on the Riemann's surface for the function

$$f(u) = u^{b-1}(1-u)^{c-b-1}(1-xu)^{-a}.$$

1. Double Circuits. The path of integration that we shall employ is a closed double circuit about two of the branch points of the function to be integrated, and is so constructed that the variable u in describing the complete path shall go just as many times about a branch point in the positive direction as it does in the negative direction, thus making the function return to its original value. Consider, for example, the double circuit integral about the points 0 and 1. If the variable u start from the point A (Fig. 1), there is first a positive circuit about 0, by which the function $f(u)$ is multiplied by the constant $e^{2\pi i(b-1)}$, since the factor u^{b-1} is multiplied by $e^{2\pi i(b-1)}$ and the other factors are unaltered. Next comes a circuit in the positive direction about the point 1, which gives rise to the factor $e^{2\pi i(c-b-1)}$. The circuits in the negative direction about 0 and 1 give rise to the factors $e^{-2\pi i(b-1)}$ and $e^{-2\pi i(c-b-1)}$ respectively, and since the product of all these factors is unity, the function $f(u)$ has reproduced itself when the variable u has returned to the point A.

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* Picard, *Traité d'Analyse*, vol. 3, p. 274.

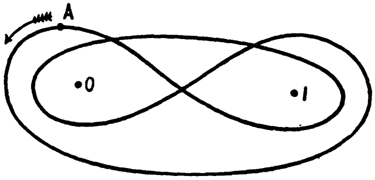

FIG. 1.

A branch point u_0 of a function $\phi(u)$ is said to have the *multiplier* ρ if, when u describes a small closed path in the positive direction about u_0, the branch of the function is thus multiplied by the factor ρ. This will, in particular, be the case if $\phi(u) = (u - u_0)^a \psi(u)$, where $\psi(u)$ is analytic at u_0 and does not vanish there. In this case a is called the *exponent* of u_0; the multiplier is $e^{2\pi i a}$. By the multiplier or exponent of the point ∞ is meant the multiplier or exponent of the point $u' = 0$, where $u' = 1/u$. Double circuits about two branch points H and K of any function will yield closed paths on the Riemann's surface of that function whenever these branch points have exponents.

Double circuit integrals were introduced by C. Jordan,* and were employed later by Nekrassoff† and Pochhammer‡. Schellenberg,§ following suggestions of Klein's, studied systematically the hypergeometric function as defined by a double circuit integral. The notion of the loop integral goes back to Riemann.‖

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* *Cours d'Analyse*, vol. 3, 1° ed., 1887, p. 241.

† *Math. Annalen*, vol. 38 (1891), p. 509.

‡ *Math. Annalen*, vols. 35, p. 470; 36, p. 84; 37, p. 500. Pochhammer discovered these integrals independently, *l. c.*, vol. 37, p. 501.

§ *Neue Behandlung der hypergeometrischen Function auf Grund ihrer Definition durch das bestimmte Integral*, Inaugural-Dissertation, Göttingen, 1892. Klein treated this function at length in a course of lectures held at Göttingen in the winter semester of 1893–94: *Die hypergeometrische Function*, Göttingen, 1894 (lithogr.; now for sale by B. G. Teubner, Leipzig).

‖ *Cf.* Klein's historical note, *Math. Annalen*, vol. 38 (1891), p. 148.

The function $f(u)$, whose integral is here to be considered, has the four branch points 0, 1, $1/x$, ∞, and thus gives rise to a variety of double circuits. It is the purpose of this paper to consider the change in the corresponding double circuit integrals when the variable branch point $1/x$ makes a closed circuit in the positive direction about one of the other three fixed branch points 0, 1, ∞. This problem in a more general form has been taken up by Schlesinger,* but it is of interest to determine directly what these changes are for the particular case of the hypergeometric function, as it is difficult to extract from his general formulæ the special results and the fundamental principle. Klein† has also considered the change in the hypergeometric function introduced when the variable branch point $1/x$ makes a closed circuit about another branch point. But he makes use of straight line integrals between two of the branch points instead of double circuit integrals. The change in the straight line integrals having been determined, the corresponding change in the double circuit integrals may be found from the relations connecting these two classes of integrals. In this paper, however, the double circuit integrals will be directly investigated.

Consider a double circuit about two branch points, H and K (Fig. 2). Evidently that path is equivalent to a succession of single loops drawn from an ordinary point P of the plane about the points H and K. That is, the double circuit is equivalent to the path formed by making first a loop from P in the positive direction around H, then a loop from P in the positive direction around K, next a loop in the negative direction around H followed by a loop in the negative direction around K. Designate by s_H a loop drawn from P in the positive direction around H, and by s_H^{-1} a loop drawn in the negative

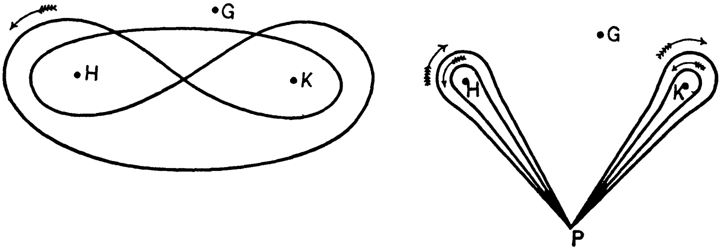

FIG. 2.

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* *Handbuch der Theorie der linearen Differentialgleichungen*, vol. 2_1, p. 427.

† *Vorlesung über die hypergeometrische Function*, p. 100.

direction. Then an integral taken along the double circuit is equivalent to an integral taken along the path $s_H s_K s_H^{-1} s_K^{-1}$, where the order of the factors from left to right in the product indicates the order in which the circuits are to be made. This is only one of an indefinite number of double circuits that can be drawn about the points H, K. Thus another double circuit is that shown in Fig. 3, where G is another one of the branch points.

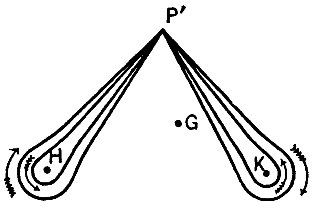

FIG. 3.

If $1/x$ make a circuit about H or K, a double circuit about H and K will be altered. We shall study the changes thus arising by means of the component loops s_H, s_K, s_H^{-1}, s_K^{-1}.

2. The Transformation of the Loops, when $1/x$ describes a Closed Path. We proceed to consider the changes introduced in the loops s_0, s_1, s_x, s_∞, drawn from a point P of the axis of reals intermediate between the points 0, 1 and encircling respectively the four branch points 0, 1, $1/x$, ∞ of $f(u)$, when the variable point $1/x$ makes a positive circuit about the point 0. The loops are to follow each other as they emerge from the point P in the order indicated in Fig. 4.* The point 0 is an obstruction over which no part of the loop s_x can pass, else the path of integration would go through a singular point. Similarly the path s_0 is an obstruction over which the point

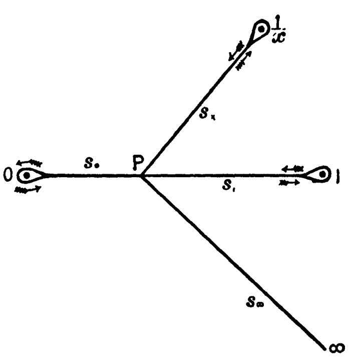

FIG. 4.

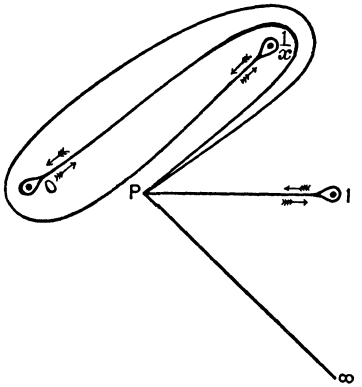

FIG. 5.

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* We remark that we get a different set of loops (one not deformable continuously into the present set) if we allow the four loops to emanate from the point P in a different order.

$1/x$ cannot pass. We may avoid these difficulties and allow $1/x$ to make the complete circuit and come back to its starting point if we imagine the loops s_0, s_x, s_1, s_∞, to be flexible cords fastened at P and stretched around the points 0, $1/x$, 1, ∞. Then after $1/x$ has made a circuit in the positive sense about 0, the position of the loops is indicated in the accompanying figure (Fig. 5).

If the transformed loop is denoted by $s^{(0)}$, then we have immediately, by inspection of the figure, the following relations:

$$(1)\qquad \begin{cases} s_0^{(0)} = s_x s_0 s_x^{-1} \\ s_x^{(0)} = s_x s_0 s_x s_0^{-1} s_x^{-1} \\ s_1^{(0)} = s_1 \\ s_\infty^{(0)} = s_\infty . \end{cases}$$

In a similar manner it is shown that, when the point $1/x$ makes a positive circuit about the point 1, the following transformation results:

$$(2)\qquad \begin{cases} s_0^{(1)} = s_0 \\ s_x^{(1)} = s_1 s_x s_1^{-1} \\ s_1^{(1)} = s_1 s_x s_1 s_x^{-1} s_1^{-1} \\ s_\infty^{(1)} = s_\infty , \end{cases}$$

where $s_H^{(1)}$ denotes the transformed loop s_H when $1/x$ makes a positive circuit around the point $u = 1$.

3. The Corresponding Transformation of the Simple Loop Integrals. We shall now investigate the changes in the fundamental simple loop integrals when $1/x$ makes a circuit about 0 or 1.

Denoting the integral $\int f(u)du$ taken along the loop s by $\int_s$, we find:

$$(3)\qquad \int_{s_0^{(0)}} = \int_{s_x s_0 s_x^{-1}} = \int_{s_x} + e^{2\pi i\delta}\int_{s_0} + e^{2\pi i(\alpha+\delta)}\int_{s_x^{-1}} .$$

Here and in what follows we have replaced $b-1$, $c-b-1$, $-a$ respectively by α, β, δ, so that the integrand becomes

$$f(u) = u^{\alpha}(1-u)^{\beta}(1-xu)^{\delta}.$$

The exponent γ of the point ∞ has the value:

$$\gamma = -\alpha - \beta - \delta - 2.$$

If α_H denote the exponent of the branch point H it is easily seen that

$$(4)\qquad \int_{s_H^{-1}} + e^{-2\pi i\alpha_H}\int_{s_H} = 0.$$

Reducing (3) by means of (4) we get:

$$\int_{s_0^{(0)}} = (1 - e^{2\pi i\alpha}) \int_{s_x} + e^{2\pi i\delta} \int_{s_0} \tag{5a}$$

Similarly,

$$\left\{\begin{aligned} \int_{s_x^{(0)}} &= (1 + e^{2\pi i(\alpha+\delta)} - e^{2\pi i\delta}) \int_{s_x} + e^{2\pi i\delta}(1 - e^{2\pi i\delta}) \int_{s_0} \\ \int_{s_1^{(1)}} &= \int_{s_1} \\ \int_{s_\infty^{(0)}} &= \int_{s_\infty} \end{aligned}\right. \tag{5b}$$

Corresponding to the circuit of the point $1/x$ about the point 1, we have the following transformation.

$$\left\{\begin{aligned} \int_{s_0^{(1)}} &= \int_{s_0} \\ \int_{s_x^{(1)}} &= (1 - e^{2\pi i\delta}) \int_{s_1} + e^{2\pi i\beta} \int_{s_x} \\ \int_{s_1^{(1)}} &= (1 + e^{2\pi i(\beta+\delta)} - e^{2\pi i\beta}) \int_{s_1} + e^{2\pi i\beta} (1 - e^{2\pi i\beta}) \int_{s_x} \\ \int_{s_\infty^{(1)}} &= \int_{s_\infty} . \end{aligned}\right. \tag{6}$$

4. Double Circuit Integrals expressed in Terms of Simple Loop Integrals. Let L_{HK} denote generally a double circuit integral about the two branch points H, K, corresponding to the loops s_H, s_K, so that

$$L_{HK} = \int_{s_H\, s_K\, s_H^{-1}\, s_K^{-1}} = \int_{s_H} + e^{2\pi i\alpha_H} \int_{s_K} + e^{2\pi i(\alpha_H+\alpha_K)} \int_{s_H^{-1}} + e^{2\pi i\alpha_K} \int_{s_K^{-1}},$$

and hence, by use of (4),

$$L_{HK} = (1 - e^{2\pi i\alpha_K}) \int_{s_H} - (1 - e^{2\pi i\alpha_H}) \int_{s_K} \tag{7}$$

In particular,

$$\left\{\begin{aligned} L_{01} &= (1 - e^{2\pi i\beta}) \int_{s_0} - (1 - e^{2\pi i\alpha}) \int_{s_1} \\ L_{0x} &= (1 - e^{2\pi i\delta}) \int_{s_0} - (1 - e^{2\pi i\alpha}) \int_{s_x} \\ L_{1x} &= (1 - e^{2\pi i\delta}) \int_{s_1} - (1 - e^{2\pi i\beta}) \int_{s_x} \end{aligned}\right. \tag{8}$$

These double circuit integrals are not linearly independent, but all can be expressed as integral linear homogeneous functions with constant coefficients of any two linearly independent ones. This follows from the fact that these integrals are all solutions of a linear differential equation of the second order. We can obtain this, however, independently by means of the component simple loop integrals, and, what is more, determine in a very simple manner the coefficients of the linear relations. We begin by showing that L_{01} can be expressed linearly in terms of L_{0x} and L_{1x} :—*

$$L_{01} = AL_{0x} + BL_{1x}.$$

Express L_{01}, L_{0x}, L_{1x} in terms of the loop integrals by means of (8). Then A and B are so to be determined that the equation

$$(1 - e^{2\pi i\beta}) \int_{s_0} - (1 - e^{2\pi i\alpha}) \int_{s_1}$$

$$= A(1 - e^{2\pi i\delta}) \int_{s_0} - \left[A(1 - e^{2\pi i\alpha}) + B(1 - e^{2\pi i\beta})\right] \int_{s_x} + B(1 - e^{2\pi i\delta}) \int_{s_1}$$

is satisfied. A sufficient condition for this is the following:

$$\begin{aligned} A(1 - e^{2\pi i\delta}) &= (1 - e^{2\pi i\beta}) \\ B(1 - e^{2\pi i\delta}) &= -(1 - e^{2\pi i\alpha}) \\ A(1 - e^{2\pi i\alpha}) + B(1 - e^{2\pi i\beta}) &= 0. \end{aligned}$$

Here the third equation for A and B is a consequence of the other two. Hence when δ is not an integer,

$$(9) \qquad L_{01} = \frac{1 - e^{2\pi i\beta}}{1 - e^{2\pi i\delta}} L_{0x} - \frac{1 - e^{2\pi i\alpha}}{1 - e^{2\pi i\delta}} L_{1x} \,.$$

In a similar manner it is shown that

$$(10) \quad \begin{cases} L_{0\infty} = \dfrac{e^{-2\pi i\alpha} - e^{-2\pi i(\alpha+\beta+\delta)}}{1 - e^{2\pi i\delta}} L_{0x} + \dfrac{e^{-2\pi i(\alpha+\beta+\delta)}(1 - e^{2\pi i\alpha})}{1 - e^{2\pi i\delta}} L_{1x} \\ L_{1\infty} = \dfrac{e^{-2\pi i\alpha}(1 - e^{2\pi i\beta})}{1 - e^{2\pi i\delta}} L_{0x} + \dfrac{1 - e^{-2\pi i(\alpha+\delta)}}{1 - e^{2\pi i\delta}} L_{1x} \\ L_{x\infty} = e^{-2\pi i\alpha} L_{0x} + e^{-2\pi i(\alpha+\beta+\delta)} L_{1x} \,. \end{cases}$$

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*That L_{0x}, L_{1x} do not, in general, vanish identically, can be seen from formulas (8) by giving to α, β, δ properly chosen special values.

5. Transformation of Double Circuit Integrals. Let us finally consider the changes in the fundamental double circuit integrals when the point $1/x$ describes a closed circuit about the point 0. We have by equation (8)

$$L^{(0)}_{0x} = (1 - e^{2\pi i\delta}) \int_{s_0^{(0)}} - (1 - e^{2\pi i\alpha}) \int_{s_x^{(0)}}$$

and substituting the values of $\int_{s_0^{(0)}}$ and $\int_{s_x^{(0)}}$ given in equations (5*a*) and (5*b*), and combining terms

$$L^{(0)}_{0x} = e^{2\pi i(\alpha+\delta)} \left[(1 - e^{2\pi i\delta}) \int_{s_0} - (1 - e^{2\pi i\alpha}) \int_{s_x} \right]$$

or by equation (8)

(11*a*)
$$L^{(0)}_{0x} = e^{2\pi i(\alpha+\delta)} L_{0x}.$$

In the same way it is shown that

(11*b*)
$$L^{(0)}_{1x} = -(1 - e^{2\pi i\beta})\, e^{2\pi i\delta} L_{0x} + L_{1x}.$$

The transformations of other double circuit integrals may be obtained in a similar manner. Thus

(12)
$$\begin{cases} L^{(1)}_{0x} = L_{0x} - (1 - e^{2\pi i\alpha}) L_{1x} \\ L^{(1)}_{1x} = e^{2\pi i(\beta+\delta)} L_{1x}. \end{cases}$$

6. The Monodromic Group. If A, B and C represent the respective transformations which L_{0x} and L_{1x} undergo when $1/x$ makes positive circuits about 0, 1 and ∞, then by equations (11)

$$A = \begin{pmatrix} e^{2\pi i(b-a)}, & 0 \\ -e^{-2\pi i a}(1 - e^{2\pi i(c-b)}), & 1 \end{pmatrix}$$

Similarly, from equations (12)

$$B = \begin{pmatrix} 1, & -(1 - e^{2\pi i b}) \\ 0, & e^{2\pi i(c-b-a)} \end{pmatrix}$$

Lastly

$$C = A^{-1} B^{-1}$$

These results may be stated in the following theorem.

The integrals L_{0x} *and* L_{1x} *are functions which are analytic everywhere except in the vicinity of the points* 0, 1 *and* ∞, *and which undergo the transformations* A, B *and* C, *when the variable* $1/x$ *makes circuits about the points* 0, 1 *and* ∞ *in the positive direction.*

In the case of the hypergeometric series the quantities a, b, c are constants. *They may, however, be replaced by single valued, or even multiple valued functions of* x (under obvious restrictions) *and the method of this paper will still remain valid.* The double circuit will continue to represent a closed path on the Riemann's surface of the function $f(u)$ and all the formulas of transformation will remain valid. But the integrals L_{0x}, L_{1x} would not be hypergeometric functions in the more general case. This generalization is due to Dr. Wylczynski.[*]

UNIVERSITY OF CALIFORNIA, BERKELEY, MAY 1900.

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[*] *Amer. Jour. of Math.*, vol. 21 (1899), p. 100.